APPROXIMATE SOLUTION TO THE PROBLEM OF HEATING BULKY BODIES AT A VARIABLE WATER EQUIVALENT OF THE GASES

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An approximate solution is obtained to the problem of heat conduction during the heating of bulky bodies in a flow bed with a linearly variable water equivalent of the gases.

For a linearly variable water equivalent of the gases

$$w_G = a_0 + a_1 z$$

without heat sources in the gaseous phase, the problem is formulated as follows [1]:

$$\pm \frac{\partial t(\rho, Z)}{\partial Z} = \frac{\partial^2 t(\rho, Z)}{\partial \rho^2} + \frac{2\nu + 1}{\rho} \cdot \frac{\partial t(\rho, Z)}{\partial \rho}, \tag{1}$$

$$-\frac{\partial t(1, Z)}{\partial \rho} + \mathrm{Bi}\,[t_G(Z) - t(1, Z)] = 0, \quad \frac{\partial t(0, Z)}{\partial \rho} = 0, \tag{2}$$

$$\left(\frac{m_{01}}{2\nu + 2} + \beta Z\right) \frac{dt_G(Z)}{dZ} = \beta\,[t_G^0 - t_G(Z)] - \mathrm{Bi}\,[t_G(Z) - t(1, Z)] - \mathrm{Bi}_1[t_G(Z) - t_a], \tag{3}$$

$$t(\rho, 0) = t_0(\rho), \quad t_G(0) = t_G^{н}, \tag{4}$$

$$Z = \frac{az}{vR^2}\; m_{01}, = \frac{a_0}{Gc}, \quad \beta = m_1 \mathrm{Bi}, \quad m_1 = \frac{a_1}{\alpha f}, \quad \mathrm{Bi} = \frac{\alpha R}{\lambda}.$$

The upper sign applies to parallel flow, the lower sign applies to counterflow, and the z-axis is positive in the direction of the gas flow.

A solution to such a problem was obtained in [1] in the form of a series by the method of the integral heat balance [2]. The convergence of this series in [1] has been proved for the case $m_{01} = 0$. According to numerical calculations, with $m_{01} > 0$ the series in [1] either converges slowly or diverges altogether. For this reason, the problem (1)-(4) will be solved here by a somewhat different method.

Following the procedure based on the integral heat balance, we multiply each term in (1) by $(2\nu + 2)\rho^{2\nu+1}$ and then integrate over the entire body thickness. A simultaneous solution of the resulting equation and the boundary condition (2) yields

$$\pm \frac{dt_{av}(Z)}{dZ} = (2\nu + 2)\,\mathrm{Bi}\,[t_G(Z) - t_s(Z)], \tag{5}$$

$$t_s = t(1, Z), \quad t_{av}(Z) = (2\nu + 2) \int_0^1 \rho^{2\nu+1} t(\rho, Z)\, d\rho. \tag{5a}$$

in place of (1) and (2).

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TABLE 1. Effect of Exponent n on the Value of ψ

Bi	Plate			Cylinder			Sphere		
	n								
	1,5	2,0	2,5	1,5	2,0	2,5	1,5	2,0	2,5
0,5	0,834	0,86	0,875	0,875	0,89	0,90	0,90	0,91	0,918
0,75	0,77	0,80	0,825	0,825	0,834	0,858	0,858	0,87	0,88
1,0	0,715	0,75	0,778	0,778	0,80	0,820	0,820	0,835	0,846
1,5	0,625	0,667	0,70	0,70	0,728	0,75	0,75	0,77	0,785
2,5	0,50	0,545	0,585	0,585	0,615	0,643	0,643	0,667	0,688
3,0	0,455	0,50	0,538	0,538	0,571	0,60	0,60	0,625	0,648

We now express t_s in terms of t_m, with the temperature distribution approximated by an n-th degree parabola:

$$t = t_s - \Delta t_m(1 - \rho^n), \quad \Delta t_m = t_s - t_c. \tag{6}$$

From (6) and (5a) we have

$$t_m = t_s - \frac{n\Delta t_m}{2\nu + 2 + n}. \tag{7}$$

Inserting (6) into (2), we obtain

$$n\Delta t_m = \mathrm{Bi}\,(t_G - t_s). \tag{8}$$

From (7) and (8) we determine t_s:

$$t_s = \frac{\mathrm{Bi}\,t_G + (2\nu + 2 + n)\,t_m}{2\nu + 2 + n + \mathrm{Bi}}, \tag{9}$$

which yields

$$t_G - t_s = \frac{2\nu + 2 + n}{2\nu + 2 + n + \mathrm{Bi}}\,(t_G - t_m). \tag{10}$$

From (10) we then obtain the criterial number which defines the uniformity of the temperature field ψ (according to G. M. Kondrat'ev's thesis [3]):

$$\psi = \frac{t_G - t_s}{t_G - t_m} = \frac{2\nu + 2 + n}{2\nu + 2 + n + \mathrm{Bi}}. \tag{11}$$

With (5) and (11) we replace (1)-(3) by

$$\pm \frac{dt_m}{dZ} = (2\nu + 2)\,\mathrm{Bi}\,\psi\,(t_G - t_m), \tag{12}$$

$$\left(\frac{m_{01}}{2\nu + 2} + \beta Z\right)\frac{dt_G}{dZ} = \beta\,(t_G^0 - t_G) - \mathrm{Bi}\,\psi\,(t_G - t_m) - \mathrm{Bi}_1(t_G - t_a). \tag{13}$$

To Eqs. (12) and (13) must be added the initial conditions at Z = 0 with (4) taken into consideration:

$$t_m(0) = t_m^{н}, \quad \pm \left.\frac{dt_m}{dZ}\right|_{Z=0} = (2\nu + 2)\,\mathrm{Bi}\,\psi(t_G^{н} - t_m^{н}). \tag{14}$$

In this way, we have obtained a system of ordinary differential equations which is analogous to the system obtained earlier in [4] for the heating of thin bodies under the same conditions.

The idea of approximately describing the heating of bulky bodies in terms of known relations for the heating of thin bodies was first proposed by V. I. Kitaev [5], who introduced the concept of the gross heat transfer coefficient

$$\alpha_2 = \alpha\psi, \quad \psi = \varphi/(\varphi + \mathrm{Bi}),$$

with factor φ depending on the body shape. He determined the value of ψ by comparing the lengths of time necessary to heat thin and bulky bodies in a parallel-flow and in a counterflow system, with the water equivalents of the gases and of the heated materials remaining in a constant ratio.

This method was developed further in [6-8]. The value of ψ was obtained in [6] with the temperature distribution in the body assumed parabolic, and in [7] on the basis of known temperature distribution

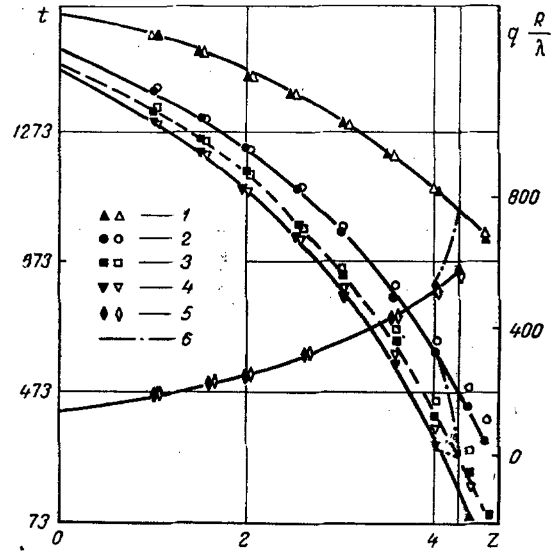

Fig. 1. Calculation of a plate heating in a counterflow system, $\beta = 0.25$, Bi = 1, $Bi_1 = 0.02$, n = 2: temperature of the gases (1), temperature of the surface (2), mean-over-the-mass temperature (3), mean-along-the-axis temperature (4), qR/λ (5), suggested adjustment of the respective values for the initial heating period Z < 0.3; solid lines represent the formulas in [1], black dots represent formulas (15)-(16), blank dots represent formulas (20)-(21).

functions in the regular mode. In [8] the ψ-number was treated according to G. M. Kondrat'ev [3] and was determined either on the basis of solutions to the equation of heat conduction or approximately on the basis of physical considerations.

No values of n for our problem are available in the literature, but one may use the data in [9], where the values of n are given for the heating of bodies at a constant water equivalent of the gases. It has been established in [10, 11] that such values of n satisfy not only the equality of temperature gradients in the exact and in the approximate solution, but also describe well the temperature distribution across a body section in the cases which have been analyzed.

It is to be noted that at medium values of the Biot number the number is not too sensitive to errors in the exponent n, especially in the case of a cylinder and a sphere (Table 1).

Assuming that ψ = const. within a particular time interval and taking into account (14), we obtain the solution to system (12)-(13) in the following form:

$$\frac{t_m(Z) - t_{G,c}}{t^{\text{н}}_m - t_{G,c}} = C_1\,{}_1F_1(1 + b_2;\ 1 + b_1 + b_2;\ \mp b_1 u)$$

$$+ C_2\left[1 + (2\nu + 2)\frac{\beta Z}{m_{01}}\right]^{-(b_1+b_2)} {}_1F_1(1 - b_1;\ 1 - b_1 - b_2;\ \mp b_1 u),$$

$$\frac{t_G(Z) - t_{G,c}}{t^{\text{н}}_m - t_{G,c}} = C_1\,[{}_1F_1(1 + b_2;\ 1 + b_1 + b_2;\ \mp b_1 u) - \frac{1 + b_2}{1 + b_1 + b_2}$$

$$\times\,{}_1F_1(2 + b_2;\ 2 + b_1 + b_2;\ \mp b_1 u)] + C_2\left[1 + (2\nu + 2)\frac{\beta Z}{m_{01}}\right]^{-(b_1+b_2)}$$

$$\times\left[\left(1 \mp \frac{b_1 + b_2}{b_1 u}\right){}_1F_1(1 - b_1;\ 1 - b_1 - b_2;\ \mp b_1 u) - \frac{1 - b_1}{1 - b_1 - b_2}\,{}_1F_1(2 - b_1;\ 2 - b_1 - b_2;\ \mp b_1 u)\right)\Bigg], \tag{15}$$

$$b_1 = \frac{\mathrm{Bi}}{\beta}\psi,\ b_2 = \frac{\mathrm{Bi}_1}{\beta},\ t_{G,c} = \frac{\beta t^0_G + \mathrm{Bi}_1 t_a}{\beta + \mathrm{Bi}_1},$$

$$u = (2\nu + 2)\left(\frac{m_{01}}{2\nu + 2} + \beta Z\right),\ C_i = \frac{C^0_i}{C}\ (i = 1,\ 2), \tag{16}$$

$$C^0_1 = \frac{1 - b_1}{1 - b_1 - b_2}\,{}_1F_1(2 - b_1;\ 2 - b_1 - b_2;\ \mp b_1 u_0) - \left(\frac{t^{\text{н}}_m - t^{\text{н}}_G}{t^{\text{н}}_m - t_{G,c}} \mp \frac{b_1 + b_2}{b_1 u_0}\right){}_1F_1(1 - b_1;\ 1 - b_1 - b_2;\ \mp b_1 u_0),$$

$$C^0_2 = \frac{t^{\text{н}}_M - t^{\text{н}}_G}{t^{\text{н}}_m - t_{G,c}}\,{}_1F_1(1 + b_2;\ 1 + b_1 + b_2;\ \mp b_1 u_0) - \frac{1 + b_2}{1 + b_1 + b_2}$$

$$\times\,{}_1F_1(2 + b_2;\ 2 + b_1 + b_2;\ \mp b_1 u_0),\ u_0 = u|_{Z=0} = m_{01},$$

$$C = {}_1F_1(1 + b_2;\ 1 + b_1 + b_2;\ \mp b_1 u_0)\left[\frac{1 - b_1}{1 - b_1 - b_2}\,{}_1F_1(2 - b_1;\ 2 - b_1 - b_2;\right. \tag{17}$$

$$\mp b_1 u_0) \pm \frac{b_1 + b_2}{b_1 u_0}\,{}_1F_1(1 - b_1;\ 1 - b_1 - b_2;\ \mp b_1 u_0)]$$

$$- {}_1F_1(1 - b_1;\ 1 - b_1 - b_2;\ \mp b_1 u_0)\frac{1 + b_2}{1 + b_1 + b_2}\,{}_1F_1(2 + b_2;\ 2 + b_1 + b_2;\ \mp b_1 u_0).$$

When $m_{01} = 0$, (15) and (16) simplify considerably:

$$\frac{t_m(Z)-t_{G,c}}{t_m^н - t_{G,c}} = {}_1F_1(1+b_2;\ 1+b_1+b_2;\ \mp u_1),\ u_1 = (2\nu+2)\,\mathrm{Bi}\,\psi Z, \tag{18}$$

$$\frac{t_G(Z)-t_{G,c}}{t_m^н - t_{G,c}} = \frac{t_m(Z)-t_{G,c}}{t_m^н - t_{G,c}} - \frac{1+b_2}{1+b_1+b_2}\,{}_1F_1(2+b_2;\ 2+b_1+b_2;\ \mp u_1). \tag{19}$$

A comparison of (15) and (16) with the test data in [12] shows that, when $m_{01} > 1$, the consideration of heat losses in the working chamber (Bi > 0) complicates the design formulas appreciably.

An analysis will also show that the heat losses affect the maximum calculated temperature of the gases $t_{G,c}$ (with $\mathrm{Bi}_1 \to 0$, $t_{G,c} \to t_G^0$) as well as the function describing the dynamics of body heating (right-hand sides of (15) and (16)).

The expressions for $t_{G,c}$ include the sum $(\beta + \mathrm{Bi}_1)$ and, therefore, at small values of β in many practical situations Bi_1 has an appreciable effect on $t_{G,c}$ and $t_m(Z)$. On the right-hand sides of (15) and (16) there appears the sum $(\mathrm{Bi}\psi + \mathrm{Bi}_1)/\beta = (b_1 + b_2)$ where Bi_1 can vary often be disregarded in comparison with $\mathrm{Bi}\psi$ (or unity).

Then considering that $b_2 = 0$ on the right-hand sides of (15) and (16), we have

$$\frac{t_m(Z)-t_{G,c}}{t_m^н - t_{G,c}} = (1 \pm m_{01}\theta_G^н)\,{}_1F_1(1;\ 1+b_1;\ \mp b_1u) + [1-(1 \pm m_{01}\theta_G^н)$$

$$\times\, {}_1F_1(1;\ 1+b_1;\ \mp b_1 m_{01})\left(\frac{u}{m_{01}}\right)^{-b_1}\exp[\mp(2\nu+2)\,\mathrm{Bi}\psi Z], \tag{20}$$

$$\frac{t_G(Z)-t_{G,c}}{t_m^н - t_{G,c}} = \frac{1}{u}\Big\{\pm(1 \pm m_{01}\theta_G^н)[1-{}_1F_1(1;\ 1+b_1;\ \mp b_1u)] \mp [1$$

$$-(1 \pm m_{01}\theta_G^н)\,{}_1F_1(1;\ 1+b_1;\ \mp b_1m_{01})]\left(\frac{u}{m_{01}}\right)^{-b_1}\exp[\mp(2\nu+2)\,\mathrm{Bi}\,\psi Z]\Big\}, \tag{21}$$

$$\theta_G^н = \frac{t_G^н - t_{G,c}}{t_m^н - t_{G,c}}.$$

If, in addition, $m_{01} = 0$, then

$$\frac{t_m(Z)-t_{G,c}}{t_m^н - t_{G,c}} = {}_1F_1(1;\ 1+b_1;\ \mp u_1), \tag{22}$$

$$\frac{t_G(Z)-t_{G,c}}{t_m^н - t_{G,c}} = \frac{b_1}{1+b_1}\,{}_1F_1(1;\ 2+b_1;\ \mp u_1). \tag{23}$$

Values of ${}_1F_1(a,\ b,\ x)$ are given in [1, 12].*

If $t_G(Z)$ and $t_m(Z)$ are known, then $t_s(Z)$ is determined from (9), while $t(0,\ Z) = t_c(Z)$ and Δt_m are calculated by the formulas

$$t_c = t_m - \frac{2\nu+2}{n}(t_s - t_m),\ \Delta t_m = \frac{\mathrm{Bi}\,\psi}{n}(t_G - t_m). \tag{24}$$

As is well known [13], function ${}_1F_1(a,\ b,\ x)$ converges for every $|x| < \infty$ and $b \neq 0, -1, -2, \ldots$ Therefore, Eqs. (15) and (16) represent the sought solutions when $(b_1 + b_2) \neq 1, 2, 3, \ldots$ Expressions (18)-(23), on the other hand, are valid for all values of the parameters.

The graph in Fig. 1, which has been copies from [1], shows also results of calculations based on (15)-(16) and (20)-(21) with (9) and (24) taken into account. The initial conditions for calculations according to the formulas derived here have also been taken from [1] for Z = 1 (in order to satisfy the condition $m_{01} > 0$ when the series in [1] is divergent). Calculations have shown that all temperature values based on the method in [1] agree with those based on (15) and (16) within 2-5°K; formulas (20)-(21) yield often an insignificant error and are entirely suitable for engineering applications (Fig. 1).

We note that the error of these approximate solutions is due mainly to the inaccuracy of n, and during the irregular heating period also due to the imprecise maintenance of the initial temperature distribution.

*Other sources where ${}_1F_1(a,\ b,\ x)$ can be found are listed in [13].

An analysis has shown [14] that under actual conditions with $n = 2$, $\beta = 0$, $m_{01} = 0.5$-1.667, and $Bi \leq 10$ the error in determining t_m does not exceed 5%. A more refined definition of the value of n would undoubtedly improve the accuracy of calculations.

A. S. Kadinova [15] compared the values of ψ for plates on the basis of test data in the regular mode [3] with the data obtained by B. I. Kitaev [5] for $n = 2$, and found that both methods yield almost the same results up to $Bi = 11$.

These solutions have been obtained on the assumption that α = const. In the general case, however, α may be a function of $W_G(z)$ (more precisely, a function of the Reynolds number) as well as of the ambient and the body temperature.

Taking the relation $\alpha(z)$ into account does not affect the linearity of the system, but it complicates considerably the coefficients in the second-order differential equation with respect to t_m (or t_G) obtained from (12)-(13), namely the dependence of these coefficients on Z. Taking into account the temperature (t_G and t_s)-dependence, on the other hand, renders the system of equations nonlinear. In both cases it becomes more difficult to solve Eqs. (12)-(13).

At the same time, the assumption of a constant heat transfer coefficient is not the major drawback in applying the solutions also to the case of a variable heat transfer coefficient, if the calculations are made piecewise only — with the average of the extreme α values used in each interval.

If the variation of α with temperature is negligible, then it becomes easy to determine α at the beginning and at the end of a computation interval from known values of w_G. If α depends on t_G and t_s, however, then the method of successive approximations must be applied here as follows. With known values of w_G, t_G, and t_s one determines α at the beginning of a computation interval. In order to determine the end values of $\alpha(\alpha_e)$, one first assumes values for t_G and t_s, then calculates α_e, and finds the average value of α $(\bar{\alpha})$, whereupon the end values to t_G and t_s are found according to the formulas derived here. If these values do not coincide with those assumed earlier, one assumes new values for t_G and t_s and repeats the process until an agreement within the prescribed accuracy is obtained.

A practical application of this method to calculations according to our formulas as well as according to [1, 4, 12] has shown that, as a rule, one or two successive approximations suffice.

According to calculations pertaining to radiative heating of bodies at a constant or at a variable ambient temperature [16-19], the solution to the linearized problem with a strong temperature-dependence of the heat transfer coefficient will yield results almost identical to those of an exact solution, if the entire heating range is broken down into three or more intervals and computations within each of them follow the described procedure.

NOTATION

t	is the instantaneous temperature of the body;
t_G	is the instantaneous temperature of the gases;
r	is the radial coordinate of a point in the body;
z	is the distance from the beginning of the heat transfer zone (entrance in the case of parallel flow, exit in the case of counterflow) along the gas stream;
$\rho = r/R$;	
$Bi_1 = KR\Pi/\lambda f$;	
$v = (2\nu + 2)G/\gamma fR$;	
R	are the radius of cylinder or sphere, or half the plate thickness;
K	is the coefficient of heat losses in the operating volume;
f	is the heating surface of bodies distributed over a unit furnace length;
G	is the furnace efficiency;
c	is the specific heat of heated bodies;
α	is the heat transfer coefficient;
a	is the thermal diffusivity;
λ	is the thermal conductivity;
γ	is the specific weight of heated bodies;
ν	is the form factor, according to [7], equal to −0.5, 0, and 0.5 respectively for a plate, a cylinder, and a sphere;
t_a	is the ambient temperature;

t_G^0	is the temperature of combustion products entering the furnace through side burners;
t_s, t_c	are the surface temperature and center temperature of heated bodies;
${}_1F(a, b, x)$	is the degenerated hypergeometric function [13];
Π	is the equivalent perimeter of furnace cross section;
q	is the thermal flux.

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